ISRAEL JOURNAL OF MATHEMATICS 77 (1992), 83–95

ON PERIODIC COMPACT GROUPS

BY

E. I. ZELMANOV

Mathematical Institute, University of Oxford
24–29 St. Giles, Oxford OX1 3LB, England

ABSTRACT

It is proved that a periodic pro-p-group is locally finite.

In [10] V. P. Platonov conjectured that periodic compact (Hausdorff) groups are locally finite. In other words the problem in question is the Burnside conjecture for compact groups. J. S. Wilson [15] proved that (under the assumption that there are finitely many finite simple sporadic groups) it suffices to prove the above conjecture for pro-p-groups. This is what we do in this paper.

THEOREM 1: *Every periodic pro-p-group is locally finite.*

From this theorem combined with [15] and with what is already known about locally finite groups ([2,6]) there follows

THEOREM 2: *Every infinite compact group contains an infinite abelian subgroup.*

We remark that as far as Theorem 2 is concerned the reduction to pro-p-groups in [15] didn't used the classification of finite simple groups.

All periodic compact groups are totally disconnected and thus pro-finite (cf. [3]). Hence V. P. Platonov's conjecture for groups of bounded exponent is equivalent to the so-called Restricted Burnside Problem (cf. [9, 14]).*

Let G be a periodic pro-p-group. Consider the closed subsets

$$G_{(n)} = \{g \in G \mid g^{p^n} = 1\}, \quad G = UG_{(n)}.$$

* It is still not known whether all periodic compact groups are groups of bounded exponent.

Received January 4, 1991 and in revised form August 15, 1991

By Baire's Category theorem (cf. [12]) one of the subsets $G_{(n)}$ contains some neighborhood, that is $G_{(n)} \supset gH$, where H is a normal subgroup of G of finite index. For an arbitrary element $h \in H$ we have

$$(gh)^{p^n} = h^{g^{p^n-1}} h^{g^{p^n-2}} \cdots h^g h = 1 \qquad (T_{p^n})$$

where $x^y = y^{-1}xy$. In order to prove that the group G is locally finite it is sufficient to prove that the subgroup H is locally finite. Let K be a finitely generated subgroup of H such that $K^g = K$. In the work [7] of E. I. Khukhro it is shown that the class of nilpotency of a finite p-group which satisfies the identity T_p is bounded from above by a function on p and the number of generators. Thus if $n = 1$, then for all open normal subgroups $H_1 \triangleleft G$ such that $H_1^g = H_1$ the nilpotency classes of $K/K \cap H_1$ are bounded from above. Hence K is nilpotent and finite (cf. [8]).

Unfortunately the class of nilpotency of a finite p-group satisfying $T_{p^n}, n > 1$, cannot be bounded from above by a function of p^n and the number of generators (cf. [1]). Instead we prove the (rather complicated) Proposition 1 below and use it in the above arguments.

For arbitrary elements x, y of a group we denote by (x, y) their group commutator $x^{-1}y^{-1}xy$. Let G be a group with an automorphism σ such that $\sigma^{p^n} = \mathrm{Id}$. For a finite subset $X = \{x_1, \ldots, x_k\} \subseteq G$ consider the set of commutators

$$\tilde{X} = \{((\ldots(x_i, \underbrace{\sigma), \sigma), \cdots, \sigma}_{m}, m < p^n, 1 \leq i \leq k\},$$

where $(x, \sigma) = x^{-1}\sigma^{-1}x\sigma = x^{-1}x^{\sigma}$ for an arbitrary element $x \in G$. Denote by $O(X, n)$ the maximum of orders of all commutators on $\tilde{X}$ of weight $\leq n$. Here by commutators on $\tilde{X}$ we mean the elements which can be expressed from the elements of the set $\tilde{X}$ by means of the operation of commutation.

PROPOSITION 1: *There exist a sequence of functions*

$$h_2 \equiv 1, \quad h_i(k, m, t_2, \ldots, t_{i-1}), \quad 3 \leq i \leq n,$$

and a function $F(k, m, t_2, \ldots, t_n)$ such that every finite p-group which has an automorphism σ with $\sigma^{p^n} = Id$, satisfies the property (T_{p^n}) and contains a finite subset $X = \{x_1, \ldots, x_k\}$ such that $\tilde{X}$ generates G, is nilpotent of class

$\leq S.O(X,S)$ *where* $S = F(p^n \cdot k, p^n, t_2, \ldots, t_n)$, *the* t_i*'s are defined inductively:* $t_i = 2 \cdot O(X, h_i(p^n \cdot k, p^n, t_2, \ldots, t_{i-1})), 2 \leq i \leq n$.

The key to the proof of Proposition 1 lies in the theory of PI-algebras. It was shown by I. Kaplansky [5] that an associative nil algebra which satisfies a polynomial identity is locally nilpotent.

A. I. Shirshov ([13], cf. also [20]) improved this result in the following way:

Let A be an associative algebra which is generated by elements $x_1, \ldots, x_k$. Suppose that (1) A satisfies a polynomial identity of degree n, (2) every word in $x_1, \ldots, x_n$ of length $\leq n$ is nilpotent. Then A is nilpotent.

In our papers [18, 19] on the Restricted Burnside Problem we proved the following Lie version of the above assertion:

Let L be a Lie algebra which is generated by elements $x_1, \ldots, x_k$. Suppose that: (1)L satisfies the polynomial identity $\Sigma[\ldots[y, x_{\sigma(1)}], \cdots, x_{\sigma(n)}] = 0, \sigma \in S_n$; (2) there exists an integer $s \geq 1$ such that for any commutator ρ on $x_1, \ldots, x_k$ we have

$$[\cdots[L, \underbrace{\rho], \ldots, \rho]}_{s} = 0.$$

Then L is nilpotent.

In this paper we strengthen this theorem in the spirit of A. I. Shirshov:

PROPOSITION 2: *There exist a sequence of functions*

$$h_2 \equiv 1, \quad h_i(r, n, t_2, \ldots, t_{i-1}), \quad 3 \leq i \leq n,$$

and a function $F(r, n, t_2, \ldots, t_n)$ *such that each Lie algebra* L

(1) *which is generated by elements* $x_1, \ldots, x_r$;

(2) *which satisfies the identity*

$$\Sigma[\cdots[y, x_{\sigma(1)}], \ldots, x_{\sigma(n)}] = 0, \qquad \sigma \in S_n; \text{ and}$$

(3) *for every commutator* ρ *on* $x_1, \ldots, x_r$ *of weight* $\leq h_i(r, n, t_2, \ldots, t_{i-1})$ *the operator* $\mathrm{ad}(\rho) : x \to [x, \rho]$ *is nilpotent of degree* $\leq t_i, i = 2, \ldots, n$,

is nilpotent of class $\leq F(r, n, t_2, \ldots, t_n)$.

The proof is based on the following difficult theorem from [18, 19].

THEOREM ([18, 19]): *A Lie algebra which satisfies the Engel's identity*

$$[\cdots[y,\underbrace{x],\ldots,x]}_{n}=0$$

is locally nilpotent.

The reduction to the theorem above closely follows that in [17] but we outline it for the benefit of the reader.

Consider the ordered alphabet $X = \{x_1, \ldots, x_k\}$; $x_i > x_j$ for $i > j$, and assume the set of all associative words on X to be partially ordered via the lexicographical ordering. Consider also the free associative algebra Ass$[X]$ on the set of generators X. Recall that an element h of the algebra Ass$[X]$ is called a **commutator** if h can be expressed from elements of the set X by means of the operation of commutation $[x,y] = xy - yx$. We shall call an associative word u from elements of the set X **special** if there exists a commutator $[u]$ for which the leading term is the word u. For example, the word $x_3x_1x_2x_2x_1$ is special because it is the leading term in the commutator $[[x_3,x_1],[[x_2,x_1],x_2]]$.

We call an associative word w n-**partitionable** if it can be represented in the form $w = w_0u_1w_1\cdots u_nw_n$ where u_i are special words and for any nonidentical permutation $\sigma \in S_n$

$$w > w_0u_{\sigma(1)}w_1\cdots u_{\sigma(n)}w_n.$$

In [17] we proved the following analog of the celebrated A. I. Shirshov's $N(k,s,n)$-lemma (cf. [13, 20]).

LEMMA 1 ([17]): *For arbitrary positive integers k,n,m there exists an integer $H(k,n,m)$ such that any word w on X of length $H(k,n,m)$ either contains a subword u^m, u being special, or is n-partitionable.*

Now we keep proving analogs of A. I. Shirshov's lemmas, this time it will be [20, lemma 4 on p. 101].

LEMMA 2: *Let w be an associative word of length $\geq 2^{n-1}(n-2)!$ which is not representable in the form v^t, where v is a proper subword of the word w. Then the word w^{2n} is n-partitionable.*

Proof: Since w is not a power of a proper subword it involves more than one letter. Let us assume that x_k is the highest letter which occurs in w. Following A. I. Shirshov we call a word v x_k-**indecomposable** if $v = x_k\cdots x_kx_{i_1}\cdots x_{i_s}$,

$s \geq 1$, $i_t \neq k$ for $t = 1, \ldots, s$. In the set T of all x_k-indecomposable words we define the linear order: $\alpha > \beta$; $\alpha, \beta \in T$, if either $\alpha > \beta$ lexicographically or α is the beginning of β. Words in the alphabet T are called T-words. Say that two words have the same composition if each letter occurs in them the same number of times. Let T-words α, β be of the same composition in the alphabet T. It is easy to see that if α is lexicographically greater than β in the alphabet T, then α is lexicographically greater than β also in the alphabet X. In particular, a special T-word with respect to T is special also with respect to X.

Let us prove the lemma by induction on n. If $n = 2$ and the word w is not 2-partitionable itself, then $w = x_{i_1} \cdots x_{i_t}, i_1 \leq \cdots \leq i_t, i_1 < i_t$. Now

$$w^2 = (x_{i_1} \cdots x_{i_{t-1}}) x_{i_t} x_{i_1} (x_{i_2} \cdots x_{i_t})$$

is the 2-partition of w^2. Since w contains x_k there exists a cyclic permutation of the word w which turns it into the T-word v. Clearly v^{2n-1} is a subword of w^{2n}. Let

$$v = x_k^{i_1} v_1 x_k^{i_2} v_2 \cdots x_k^{i_t} v_t,$$

where $i_1, \ldots, i_t \geq 1$ and $v_1, \ldots, v_t$ are the words on $x_1, \ldots, x_{k-1}$. Suppose that some power $i_\alpha, 1 \leq \alpha \leq t$, is not less than $n-1$. Then $v = v' x_k^{n-1} x_j v'', j < k$. Let $u_1 = x_k^{n-1} x_j, u_2 = x_k^{n-2} x_j, \ldots, u_{n-1} = x_k x_j, u_n = x_j$. Now

$$v^n = v' u_1 (v'' v' x_k) u_2 (v'' v' x_k^2) u_3 \cdots u_n v''$$

is the n-partition of v^n which implies that v^{2n-1} and w^{2n} are also n-partitionable.

Suppose that the length of some word v_α is not less than $n-1$. Then $v = v' x_k x_{j_1} \cdots x_{j_{n-1}} v''$, where $1 \leq j_1, \ldots, j_{n-1} \leq k-1$. Let $u_1 = v_k, u_2 = x_k x_{j_1}, \ldots, u_n = k_k x_{j_1} \cdots x_{j_{n-1}}$. Again

$$v^n = v' u_1 (x_{j_1} \cdots x_{j_{n-1}} v'' v') u_2 (x_{j_2} \cdots x_{j_{n-1}} v'' v') u_3 \cdots u_n v''$$

is the n-partition.

Now we may assume that $1 \leq i_1, \ldots, i_t \leq n-2$ and the length of each v_σ is $\leq n-2$. Hence

$$t \geq 2^{n-2}(n-3)!.$$

By the induction assumption the word v^{2n-2} is $(n-1)$-partitionable. Let $v^{2n-2} = w_0 u_1 w_1 u_2 \cdots w_{n-1}$ be the $(n-1)$-partition, let $v = v' x_j$, $j < k$, and denote $u_n = x_j$. Then

$$v^{2n-1} = w_0 u_1 w_1 u_2 \cdots w_{n-2} u_{n-1} (w_{n-1} v') u_n$$

is the n-partition of v^{2n-1}. This proves the lemma. ∎

LEMMA 3: *Any word w on X which is not representable in the form v^t, where v is a proper subword of the word w, can be turned into a special word by a cyclic permutation.*

Proof: We shall prove the lemma by induction on the length of w. Let x_k be the highest letter which occurs in w. Then, as we have remarked earlier, some cyclic permutation turns w into a T-word w'. The T-length of w' is less than the X-length of w' so it remains to use the induction assumption.

LEMMA 4: *For arbitrary integers $k, n, m \geq 1$ there exists an integer $H'(k,n,m)$ such that every word w on X of length $H'(k,n,m)$ either contains a subword u^m, u being special of length $< 2^{n-1}(n-2)!$, or is n-partitionable.*

Proof: Let $H'(k,n,m) = H(k,n,s)$, where $s = \max(m, 2n)+1$. Let w be a word on X of length $H'(k,n,m)$. We assume that w is not n-partitionable. Then by Lemma 1 w contains a subword u^s. Let $u = v^t$ when the word v is not a power of its proper subword. Remark that though by Lemma 1 the word u might be assumed to be special, we cannot assume that about the word v. By Lemma 2 the length of v is less than $2^{n-1}(n-2)!$. Now from Lemma 3 it follows that some cyclic permutation turns v into a special word v'. Clearly v'^{s-1} is a subword of the word v^s. Since $s-1 \geq m$ we conclude that the word v'^m is a subword of the word w which finishes the proof.

LEMMA 5 ([17]): *Let A be an associative algebra and let L be a subalgebra of the commutator Lie algebra $A^{(-)}$. Suppose that: (1) A is generated by L; (2) L is generated by m elements $x_1, \ldots, x_m$; (3) L satisfies the Engel's identity*

$$[\cdots[y, \underbrace{x], \ldots, x]}_{n} = 0;$$

(4) for arbitrary element $a \in L$ we have $a^n = 0$. Then A is nilpotent.

Clearly there exists an upper bound for the nilpotency degrees of algebras A with these properties. Denote it by $g(m,n)$.

Proof of Proposition 2: We shall use induction on $q, 1 \leq q \leq n$, to construct a sequence of functions $h_2 \equiv 1, h_i(r, n, t_2, \ldots, t_{i-1}), 3 \leq i \leq n$, and to prove the following assertion:

Let L be a Lie algebra which is generated by the subset $X = \{x_1, \ldots, x_r\} \leq L$ and A an associative algebra such that the commutator Lie algebra $A^{(-)}$ contains L and A is generated by L. Assume further that:
(a) L satisfies the linearized Engel's identity

$$\Sigma[\cdots[x, y_{\sigma(1)}], \ldots, y_{\sigma(n)}] = 0, \quad \sigma \in S_n;$$

(b) for arbitrary elements $a_1, \ldots, a_q \in L$ we have

$$\Sigma a_{\sigma_1} \cdots a_{\sigma_q} = 0, \quad \sigma \in S_q;$$

(c) for an arbitrary commutator ρ on X of length $\leq h_i(r, n, t_2, \ldots, t_{i-1})$ we have

$$\rho^{t_i} = 0, \quad 2 \leq i \leq q.$$

Then A is nilpotent.

For $q = 2$ it follows from (b) that for arbitrary $x, y \in L$ we have $xy + yx = 0$. The condition (c) implies $x_i^{t_2} = 0, 1 \leq i \leq r$ (indeed, $h_2 \equiv 1$). Hence A is nilpotent of degree $\leq (t_2 - 1)r + 1$.

Now let us assume that the functions $h_2, \ldots, h_{q-1}(r, n, t_2, \ldots, t_{q-2})$ have been constructed. By induction there exists a function $d_{q-1}(r, n, t_2, \ldots, t_{q-1})$ such that any associative algebra satisfying the above conditions with parameters $r, n, q-1, t_2, \ldots, t_{q-1}$ is nilpotent of degree $\leq d_{q-1}(r, n, t_2, \ldots, t_{q-1})$. The rest of the proof follows [17] almost verbatim.

Let $k = g(r^d, n)$, where $d = d_{q-1}(r, n, t_2, \ldots, t_{q-1})$.

Define

$$h_q(r, n, t_2, \ldots, t_{q-1}) = 2^{k-1}(k-2)!.$$

We shall show that any associative algebra A satisfying (a), (b), (c) with the parameters $r, n, q, t_2, \ldots, t_q$ is nilpotent of degree $\leq N = H'(r, k, t_q)$. If $A^N \neq 0$ then A contains a word w on X of length N such that w is not a linear combination of words which are lexicographically less than w. By Lemma 4 either w contains a subword u^{t_q}, u being special of length $\leq 2^{k-1}(k-2)!$, or w is k-partitionable. Let us consider the first case and let u be the highest word in the commutator $[u]$ on X; the weight of u is $\leq 2^{k-1}(k-2)!$. By the assumption (c), $[u]^{t_q} = 0$, hence u^{t_q} is the linear combination of words which are lexicographically less than u^t and so is the word w.

Now let us suppose that $w = w_0u_1w_1\cdots u_kw_k, u_i$ is the highest word in the commutator $[u_i]$ on X and for any nonidentical permutation $\sigma \in S_k$ we have $w > w_0u_{\sigma(1)}w_1\cdots u_{\sigma(k)}w_k$.

Let P be the ground field, consider the associative P-algebra E presented by generators $e_i, i \geq 1$, and relations $e_i^2 = 0, e_ie_j = e_je_i, 1 \leq i,j \leq k, \hat{E} = E + P.1$. Consider the tensor product $A \oplus_P \hat{E}$ and the element

$$u = \sum_{i=1}^{k}[u_i] \otimes e_i \in A \otimes_P \hat{E}.$$

We shall prove that $(uA)^k = 0$ and thus $w_0uw_1u\cdots uw_k = 0$. This contradicts what we have assumed about w.

By the choice of d any word α on X of length $\geq d$ can be presented as

$$\alpha = \sum_i \alpha_i \left(\sum_{\sigma \in S_{q-1}} \alpha_{i\sigma(1)} \cdots \alpha_{i\sigma(q_1)} \right),$$

where the α_{ij} are commutators on X and the α_i are words on X.

Suppose that there exist such words $v_1, \ldots, v_{k-1}$ on X that

$$uv_1uv_2\cdots uv_{k-1}u \neq 0.$$

The sequence $v_1, \ldots, v_{k-1}$ may be assumed to have a lexicographically minimal vector of length $(d_1, \ldots, d_{k-1})$ among all sequences of words which satisfy (2). From the lexicographical minimality and (1) it follows that $d_i \leq d-1$ for all $i, 1 \leq i < k$. For a word $v = x_{i_1}\cdots x_{i_e}$ denote by $[vu]$ the commutator $[x_{i_1}, [x_{i_2}, [\cdots [x_{i_e}, u]]\cdots]$. Again by the lexicographical minimality of $(d_1, \ldots, d_{k-1})$ we have

$$uv_1u\cdots uv_{k-1}u = u[v_1u][v_3u]\cdots[v_{k-1}u] \neq 0.$$

Now remark that the Lie algebra $L \otimes_P E$ satisfies the n-Engel's identity and, besides, for an arbitrary element $a \in L \otimes_P E \subseteq A \otimes_P E$, we have $a^q = a^n = 0$. There are not more than r^d distinct elements among $u, [v_i, u] \in L \otimes_P E$. Hence by Lemma 5 and by the choice of k we have $u[v_1u]\cdots[v_{k-1}u] = 0$, a contradiction.

Now let a Lie algebra L satisfy the conditions of Proposition 2. Then the Lie algebra $\mathrm{ad}(L)$, together with the associative algebra $R(L)$ which is generated by ad(L) in $\mathrm{End}_p(L)$, satisfy the assumptions (a), (b), (c). Clearly, the class of nilpotency of L is bounded by a function $F(r, n, t_2, \ldots, t_n)$. This finishes the proof of Proposition 2. ∎

Now let G be a finite p-group with an automorphism φ such that $\varphi^{p^n} = \mathrm{Id}$ and for an arbitrary element $a \in G$ we have $a^{\varphi^{p^n-1}} \cdots a^{\varphi} a = 1$. Assume further that G contains a subset $X = \{x_1, \ldots, x_k\}$ such that the set

$$\tilde{X} = \{x_i, (\cdots(x_i, \underbrace{\varphi), \ldots, \varphi)}_{j}, 1 \le j \le p^n - 1, 1 \le i \le k\}$$

generates G.

For a commutator ρ on $x_1, \ldots, x_k, \varphi$ denote by $w_x(\rho), w_\varphi(\rho)$ the weights of ρ with respect to X and φ respectively, so $w_x(\rho) + w_\varphi(\rho)$ is the weight of the commutator ρ.

Let us consider the Cartesian product of integers $Z^2 = \{(i,j)\}$ with the lexicographical order: $(i,j) > (k,\ell)$ whenever $i > k$ or $i = k, j > \ell$. For an arbitrary pair (i,j) let G_{ij} be the subgroup of G which is generated by all commutators ρ on X, φ such that $(w_x(\rho), w_\varphi(\rho)) > (i,j)$ and by all powers ρ^{p^k} such that $(p^k w_x(\rho), p^k w_\varphi(\rho)) \ge (i,j)$. Clearly $G_\alpha \subseteq G_\beta$ for $\beta < \alpha$ and $(G_\alpha, G_\beta) \subseteq G_{\alpha+\beta}$ for arbitrary $\alpha, \beta \in Z^2$.

For $\alpha \in Z^2$ denote by $\tilde{G}_\alpha$ the subgroup of G generated by all G_β's such that $\beta > \alpha$. Abelian factors $G_\alpha/\tilde{G}_\alpha$ may be viewed as linear spaces over the finite field $Z_p, |Z_p| = p$. Following [4, 11, 16] we consider the direct sum

$$L(G) = \bigoplus_{\alpha \in Z^2} G_\alpha/\tilde{G}_\alpha.$$

Brackets $[a_\alpha \tilde{G}_\alpha, b_\beta \tilde{G}_\beta] = (a_\alpha, b_\beta)\tilde{G}_{\alpha+\beta}$, where $a_\alpha \in G_\alpha, b_\beta \in G_\beta$, define the structure of a Lie algebra on $L(G)$.

LEMMA 6:

(a) *The Lie algebra $L(G)$ satisfies the polynomial identity*

$$\Sigma[\cdots[y, x_{\sigma(1)}], \ldots, x_{\sigma(p^n)}] = 0, \quad \sigma \in S_{p^n};$$

(b) *If $a \in G_\alpha, a^{p^t} = 1, b = a\tilde{G}_\alpha \in L(G)$ then*

$$[\cdots[L(G), \underbrace{b], \ldots, b]}_{2p^t} = 0.$$

Proof: (a) Consider the free group $\tilde{F}$ on the free generators $y, x_0, x_1, \ldots$ and the normal closure $\tilde{F}_x$ of the generators $x_0, x_1, \ldots$ in $\tilde{F}$. Let H be the normal closure of the set $\{y^{p^n}, a^{y^{p^n-1}} \cdots a^y a, a \in \tilde{F}_x\}$. Denote $F \simeq \tilde{F}/H, F_x = \tilde{F}_x H/H$. By abuse of notation we shall keep denoting the cosets yH, x_iH by y, x_i respectively. Denote by C the normal closure of the element x_0 in F: C is generated by x_0 and all commutators containing x_0. Consider also the subgroup $(C,C)_p$ of C generated by commutators (a,b) and powers a^p and a,b are arbitrary elements of C. The abelian group $V = C/(C,C)_p$ is a linear space over Z_p. For an arbitrary element $x \in F$ denote by x' the linear transformation of V induced by commutation with $x, x' : a(C,C)_p \to (a,x)(C,C)$. From the well-known commutator formula $(z,xy) = (z,y)(z,x)(z,x,y)$ it follows that for arbitrary elements $x, y \in F$ we have $(xy)' = x' \circ y'$ where $a \circ b = a + b + ab$. Clearly

$$(3) \qquad (x^{p^n})' = x'^{\circ p^n} = (1+x)'^{p^n} - 1 = x'^{p^n}.$$

For an arbitrary element $x \in F_x$ we have $(yx)^{p^n} = 1$, which implies

$$0 = ((yx_1 \cdots x_{p^n})^{p^n})' = (y' \circ x_1' \circ \cdots \circ x_{p^n}')^{p^n}$$

$$(2) \qquad = [(1+y') \cdots (1+x_{p^n}') - 1]^{p^n}.$$

Let us expand the right side as the sum of monomials in the y', x_i'. By putting $x_1 = 1$ (thus $x_1' = 0$) we see that the sum of all monomials which don't contain x_1' is zero. Hence the sum of all monomials which do contain x_1' is also zero. Then in the latter sum we put $x_2 = 1$ and so on. As a result we get that the sum of all monomials containing each of $x_1', \ldots, x_{p^n}'$ is zero. The lowest degree component of that sum is

$$\sum_{\sigma \in S_{p^n}} x'_{\sigma(1)} \cdots x'_{\sigma(p^n)}.$$

Thus we proved that

$$(3) \qquad \prod_{\sigma \in S_{p^n}} (\cdots (x_0, \ldots, x_{\sigma(p^n)}) = \rho_1 \cdots \rho_u,$$

where the ρ_i are either commutators on $y, x_0, \ldots, x_{p^n}$ which involve x_0 at least twice, or p-th powers of commutators involving x_0, or commutators of degree $\geq p^n + 2$ involving each of $x_0, \ldots, x_{p^n}$.

In the same way as before (put $x_1 = 1$, then put $x_2 = 1$ and so on) we may assume all $\rho_1, \dots, \rho_u$ involve each of the generators $x_0, \dots, x_{p^n}$. Now the assertion (a) immediately follows from (5).

(b) From (1) it follows that

$$(4) \qquad (\cdots(x_0, \underbrace{x_1), \dots, x_1)}_{p^t} = (x_0, x_1^{p^t})\rho_1 \cdots \rho_i,$$

where the ρ_j are either commutators on x_0, x_1 which involves x_0 at least twice, or p-th powers of commutators involving x_0, or commutators of degree $\geq p^t + 2$. Now it suffices to put

$$(\cdots(x_0, \underbrace{x_1), \dots, x_1)}_{p^t}$$

instead of x_0 in (4) to finish the proof of the lemma.

Now let

$$t_2 = 2\cdot O(X, 1), \quad t_3 = 2\cdot O(X, h_3(kp^n, p^n, t_2)), \quad t_4 = 2\cdot O(X, h_4(kp^n, p^n, t_2, t_3))$$

$$\dots, \quad t_n = 2 \cdot O(X, h_n(kp^n, p^n, t_2, \dots, t_{n-1})).$$

By Proposition 2 the subalgebra of $L(G)$ generated by $\tilde{X}$ is nilpotent of degree $\leq s = F(kp^n, p^n, t_2, \dots, t_n)$. Let us show that the group G is nilpotent of class $\leq s \cdot O(X, s)$. To do that we shall prove that for any commutator ρ on $\tilde{X}$ we have $\rho^{p^m} = 1$ whenever $p^m w_x(\rho) > s \cdot O(X, s)$.

Since G is nilpotent it follows that if the pair $(p^m w_x(\rho), p^m w_\sigma(\rho))$ is big enough then $\rho^{p^m} = 1$. Now if $w_x(\rho) < s$ then ρ is the commutator on $\tilde{X}$ of weight $< s$ whereas $p^m > 0(\tilde{X}, s)$, so $\rho^{p^m} = 1$. If $w_x(\rho) \geq s$ then ρ^{p^m} is a product of elements $\rho_i^{p^{m_i}}$, the ρ_i being commutators on $\tilde{X}$ and

$$(p^{m_i} w_x(\rho_i), p^{m_i} w_{\sigma(p_i)}) > (p^m w_x(\rho), p^m w_\sigma(\rho)).$$

Thus $\rho^{p^m} = 1$ and Proposition 1 is proved.

Proof of Theorem 1: As we have already remarked in the beginning of the paper there exist a normal subgroup H of G of finite index, an element $g \in G$ and an integer $n \geq 1$ such that for an arbitrary element $h \in H$ we have

$$(gh)^{p^n} = h^{g^{p^n-1}} h^{g^{p^n-2}} \cdots h^g h = 1.$$

Let $X = \{x_1, \ldots, x_k\}$ be an arbitrary finite subset of H,

$$\tilde{X} = \{x_i, ((\cdots(x_i, g), \underbrace{g), \ldots, g)}_{j}, 1 \le j \le p^n - 1, 1, \le i \le k\}.$$

Since G is periodic we may consider

$$t_2 = 2 \cdot O(X, 1), \qquad t_3 = 2 \cdot O(X, h_3(kp^n, p^n m, t_2)),$$
$$\cdots, \quad t_n = 2 \cdot O(X, h_n(kp^n, p^n, t_2, \cdots, t_{n-1})).$$

Let $s = F(k \cdot p^n, p^n, t_2, \cdots, t_n)$ and consider also $0(X, s)$. Denote by K the subgroup of H generated by $\tilde{X}$. From Proposition 1 it follows that for an arbitrary normal subgroup $H_1 \lhd G$ of finite index such that $H_1^g = H_1$, the quotient group $K/K \cap H_1$ is nilpotent of class $\le s \cdot 0(X, s)$. Hence K is nilpotent of class $\le s \cdot 0(X, s)$ and finite. We proved that the subgroup H is locally finite which implies that the group G is locally finite as well.

ACKNOWLEDGEMENT: The author is grateful to Aner Shalev whose important remarks led to many revisions in the text.

References

1. D. Gorenstein, *Finite Groups*, Harper & Row, New York, 1968.
2. P. Hall and C. R. Kulatilaka, *A property of locally finite groups*, J. London Math. Soc. **39** (1964), 235–239.
3. E. Hewitt and K. Ross, *Abstract Harmonic Analysis*, Vol. II, Springer-Verlag, New York, Berlin, 1970.
4. S. A. Jennings, *The structure of the group ring of a p-group over a modular field*, Trans. Am. Math. Soc. **50** (1941), 175–185.
5. I. Kaplansky, *Rings with a polynomial identity*, Bull. Am. Math. Soc. **54** (1948), 575–580.
6. M. I. Kargapolov, *On a problem of O. Ju. Smidt*, Siber. Math. Ž. **4** (1963), 232–235.
7. E. I. Khukhro, *On locally nilpotent groups admitting a splitting automorphism of prime order*, Mat. Sb. **130**, No. 1 (1986), 120–127.
8. E. I. Khukhro, *A comment on periodic compact groups*, Siber. Math. J. **30**, No. 3 (1989), 187–190.
9. A. I. Kostrikin, *Around Burnside*, Nauka, Moscow, 1986.
10. *Kourovka Notebook* (unsolved problems in Group Theory), 9th ed., Inst. Mat. SO AN SSSR, Novosibirsk, 1984.

11. W. Magnus, *A connection between the Baker–Hausdorff formula and a problem of Burnside*, Ann. Math. **52** (1950), 11–26; *Errata*, Ann Math. **57** (1953), 606.
12. W. Rudin, *Functional Analysis,* McGraw-Hill, New York, 1973.
13. A. I. Shirshov, *On rings with identical relations*, Mat. Sb. **43** (1957), 277-283.
14. M. Vaughan-Lee, *The Restricted Burnside's Problem*, Oxford University Press, 1990.
15. J. S. Wilson, *On the structure of compact torsion groups*, Monatsch. Math. **96** (1983), 57–66.
16. H. Zassenhaus, *Ein Verfahren jeder endlichen p-gruppe einem Lie-Ring mit der characteristic p zu zuordnem*, Abh. Mat. Sem. Univ. Hamburg **13** (1940), 200–207.
17. E. I. Zelmanov, *On some problems in the theory of groups and Lie algebras*, Mat. Sb. **180**, No. 2 (1989), 159–167.
18. E. I. Zelmanov, *Solution of the restricted Burnside's problem for groups of odd exponent*, Izvestia AN SSSR **54** (1990).
19. E. I Zelmanov, *Solution of the restricted Burnside's problem for 2-groups*, Mat. Sb., to appear.
20. K. A. Ževlakov, A. M. Slin'ko, I. P. Shestakov and A. I. Shirshov, *Rings That Are Nearly Associative*, Academic Press, New York, 1982.